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Penetrability of a one-dimensional Coulomb potential

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Abstract. A few years ago the author and his collaborators introduced the concept of a Dirac oscillator and extended it to a quark–antiquark ($q\bar{q}$) system for a discussion of the mass spectrum of mesons. The problem was reduced to a radial equation of a familiar type but with a singularity $(r-a)^{-1} \equiv x^{-1}$ at some given value of the radius. The character of the mass spectrum was determined by whether the potential x^{-1} with $-\infty \leqslant x \leqslant \infty$ is penetrable at the origin or not. This leads to a discussion of penetrability of a one-dimensional Coulomb barrier which is the object of this paper. The solution of the corresponding wave equation for $x < 0$ and $x > 0$ is well known, but the trick is to join them at $x = 0$ where they are bounded but have an ∞ derivative. We obtain explicitly the transmission and reflection amplitudes. As the barrier is then penetrable, our $q\bar{q}$ system does not have a bound spectrum, but a continuous one which, in some cases, may have resonances whose widths are small compared with their separation.

1. Introduction

In 1989, in this journal [1], the concept of a one-body Dirac oscillator was introduced, and it gave rise to several papers [2]. The author and his collaborators were principally interested in generalizing the concept to many bodies, including the three quark case [3] applied to baryons and the quark–antiquark ($q\bar{q}$) system of interest in the mass spectra of mesons [4].

The latter problem, in the centre-of-mass frame, can be reduced to a radial equation [5] of the familiar Schrödinger type with potential of the form

$$V = -b^2/(r^2 - a^2) \tag{1.1}$$

where b^2, a^2 are some real positive functions of the energy and total angular momentum. It was then important to know the energy spectrum of this radial equation which depends on the nature of the singularity at $r = a$. The potential (1.1) which is $+\infty$ at $r = a - 0$, together with the $+\infty$ value of the centrifugal force at $r = 0$, implies that we have a discrete spectrum with the wavefunction restricted to the interval $0 \leqslant r \leqslant a$. If the potential (1.1) is penetrable at $r = a$ then the spectrum is clearly continuous and the wavefunction appears in the full interval $0 \leqslant r \leqslant \infty$ though, as in other barrier problems, there may be resonant states corresponding to outgoing waves at $r \to \infty$. We will show that the latter conclusion is the one that holds.

As the potential (1.1) in the vicinity of $r = a$ can be written as

$$V = -b^2/(r+a)(r-a) \simeq -b^2/2ax \tag{1.2}$$

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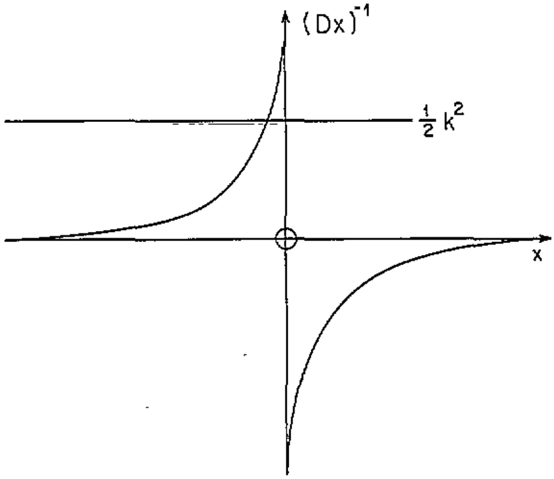

Figure 1. The potential of the one-dimensional Coulomb problem of (2.1).

where $x = (r - a)$, the important part of V is a one-dimensional Coulomb problem which leads us to a discussion of the equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - \frac{Ze^2}{x}\phi = E\phi \tag{1.3}$$

where $-\infty \leqslant x \leqslant \infty, m$ is the mass of the particle, Ze the positive charge of the one-dimensional Coulomb potential and $-e$ is that of the incoming particle.

We know full well the solutions [6] of equation (1.3) when $x < 0$ and $x > 0$ and the point is to connect them at the origin where they are bounded but their derivative is ∞. We shall indicate in the following section how this can be achieved, and obtain the transmission and reflection coefficients associated with the barrier.

We wish to stress that the one-dimensional Coulomb problem has been discussed very extensively [7], but always using $|x|^{-1}$ as the potential and not x^{-1} and, in addition, restricting the analysis to bound states and not scattering states as in this problem. Thus, as far as we know, the problem of solving equation (1.3) has not been tackled in the literature, despite its seemingly elementary nature.

We shall show in this paper that the one-dimensional Coulomb barrier is penetrable, thus affecting the nature of the spectrum of the $q\bar{q}$ system with Dirac oscillator interaction [8], as was indicated above.

2. Reduction of the problem to the Whittaker equation

Writing the energy in terms of the wave number, i.e. $E = (\hbar^2k^2/2m)$, equation (1.3) becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{2}{Dx}\phi + k^2\phi = 0 \tag{2.1}$$

where

$$D = \hbar^2(me^2Z)^{-1} \tag{2.2}$$

and the potential is drawn in figure 1.

With the help of the definitions

$$z \equiv 2ik|x| \qquad \lambda = (ikD)^{-1} \tag{2.3a, b}$$

we see that (2.1) reduces to the standard Whittaker form of the confluent hypergeometric equation [9],

$$\frac{d^2\phi^{\pm}_{(z)}}{dz} - \left(\frac{1}{4} \pm \frac{\lambda}{z}\right)\phi^{\pm}(z) = 0 \tag{2.4}$$

where the $+$ sign applies when $x > 0$ and the minus sign when $x < 0$.

As the μ appearing in equation (9.220.1) of [9] takes the value $\mu = 1/2$, we see [9] that ϕ^-, ϕ^+, can be expressed, respectively, as linear combinations of $W_{\lambda,1/2}(z), W_{-\lambda,1/2}(-z)$ or $W_{-\lambda,1/2}(z), W_{\lambda,1/2}(-z)$.

From now on we suppress the fixed index $1/2$ in $W_{\pm\lambda,1/2}(z)$ and thus refer to the linearly independent solutions of (2.4) as combination of

$$W_{\mp\lambda}(z), W_{\pm\lambda}(-z). \tag{2.5a}$$

As the asymptotic form of $W_\lambda(z)$ for $|z| \to \infty$, given by (9.227.1) of [9], is

$$W_\lambda(z) \simeq \exp(-z/2)z^\lambda \tag{2.5b}$$

we can take as solution of our problem for a wave incoming from the left as

$$\phi_\lambda^-(z) = W_\lambda(z) - R(\lambda)W_{-\lambda}(-z) \qquad \text{for } x < 0 \tag{2.6a}$$

$$\phi_\lambda^+(z) = T(\lambda)W_\lambda(-z) \qquad \text{for } x > 0 \tag{2.6b}$$

where $R(\lambda)$, $T(\lambda)$ are, respectively, the reflection and transmission coefficient for this wave.

The normal restriction that we have to put on solution (2.6) is that $\phi_\lambda^\pm(z)$ and its derivative with respect to z should be continuous at $z = 0$, i.e. $x = 0$. This is straightforward for $\phi_\lambda^\pm(0)$ but not for the derivative as $[d\phi_\lambda^\pm(z)/dz]$ diverges at $z = 0$.

Thus we proceed to discuss some characteristics of the behaviour of $W_\lambda(z)$ that allows us to avoid this problem.

3. Behaviour of the function $W_\lambda(z)$ and its derivative near $z = 0$

From equation (9.237.1) of [9] we see that $W_\lambda(z)$ admits the series expansion

$$W_\lambda(-z) = \exp(-z/2)[\Gamma(-\lambda)\Gamma(1-\lambda)]^{-1} \times\left\{\Gamma(-\lambda) - \sum_{k=0}^{\infty}\frac{\Gamma(k+1-\lambda)}{k!(k+1)!}z^{k+1}[\psi(k+1)+\psi(k+2)-\psi(k+1-\lambda)-\ln z]\right\} \tag{3.1}$$

where

$$\psi(u) = [d\ln\Gamma(u)/du]. \tag{3.2}$$

From this it is immediately clear that

$$W_\lambda(0) = [\Gamma(1-\lambda)]^{-1} \qquad W_{-\lambda}(0) = [\Gamma(1+\lambda)]^{-1} \tag{3.3a, b}$$

as $z \ln z$ also tends to zero when $z \to 0$.

Taking the derivative of $W_\lambda(z)$ it can be immediately seen that $[\mathrm{d}W_\lambda(z)/\mathrm{d}z]$ diverges as $\ln z$ when $z \to 0$, yet using the results of [9] one can show straightforwardly that [10–13]

$$\lim_{x\to 0}\left[\frac{\mathrm{d}W_\lambda(z)}{\mathrm{d}x} - 2ik\lambda \ln z W_\lambda(z)\right] = -2ikf(\lambda) \tag{3.4a}$$

$$\lim_{x\to 0}\left[\frac{\mathrm{d}W_{-\lambda}(-z)}{\mathrm{d}x} - 2ik\lambda \ln z W_{-\lambda}(-z)\right] = -2ikg(\lambda) \tag{3.4b}$$

$$\lim_{x\to 0}\left[\frac{\mathrm{d}W_\lambda(-z)}{\mathrm{d}x} - 2ik\lambda \ln z W_\lambda(-z)\right] = 2ikg(-\lambda). \tag{3.5c}$$

where

$$[\Gamma(1-\lambda)]^{-1}\{\tfrac{1}{2} - \lambda[2C + \psi(1-\lambda)]\} \equiv f(\lambda) \tag{3.5a}$$

$$-f(-\lambda) - \mathrm{i}\pi\lambda[\Gamma(1+\lambda)]^{-1} \equiv g(\lambda) \tag{3.5b}$$

and C is the Euler constant $C = 0.577\,215$.

With these results we now pass to a discussion of the boundary conditions at the origin.

4. Boundary conditions at $x = 0$

The continuity of the wavefunctions at $x = 0$ leads to the equation

$$\phi_\lambda^-(0) = \phi_\lambda^+(0) \tag{4.1}$$

and using (2.6) and (3.3) we obtain

$$[\Gamma(1-\lambda)]^{-1} - R(\lambda)[\Gamma(1+\lambda)]^{-1} = T(\lambda)[\Gamma(1-\lambda)]^{-1}. \tag{4.2}$$

For the derivative we have the divergence problem discussed in the previous section but, because of (4.1), we can write instead of the continuity of the derivative the equation

$$\left[\frac{\mathrm{d}\phi_\lambda^-(z)}{\mathrm{d}x} - 2ik\lambda \ln z \phi_\lambda^-(z)\right]_{x=0} = \left[\frac{\mathrm{d}\phi_\lambda^+(z)}{\mathrm{d}x} - 2ik\lambda \ln z \phi_\lambda^+(z)\right]_{x=0} \tag{4.3}$$

where, from the discussion in the previous section, the square brackets are perfectly definite at $x = 0$. We thus obtain from (2.6) and (3.4) the relations

$$-f(\lambda) + g(\lambda)R = g(-\lambda)T. \tag{4.4}$$

From (4.2), (4.4) we then have that

$$T = 1 - R\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)} \tag{4.5a}$$

$$R = \frac{f(\lambda) + g(-\lambda)}{g(\lambda) + g(-\lambda)\Gamma(1-\lambda)[\Gamma(1+\lambda)]^{-1}} \tag{4.5b}$$

where $f(\lambda)$ and $g(\lambda)$ are given by (3.5). Substituting in (4.5*b*) we finally obtain

$$R(\lambda) = \frac{(\mathrm{i}\pi\lambda)\Gamma(1+\lambda)}{\Gamma(1-\lambda)\{1+\lambda[\psi(1-\lambda)-\psi(1+\lambda)]\}}. \tag{4.6}$$

We can further simplify expression (4.6) by noting that from (8.365.8) and (8.365.1) in [9] we have

$$\psi(1-\lambda) = \psi(\lambda) + \pi\tan^{-1}\pi\lambda \qquad \psi(1+\lambda) = \psi(\lambda) + \lambda^{-1} \tag{4.7a, b}$$

so finally we get

$$R = \frac{\Gamma[1-(\mathrm{i}/kD)]}{\Gamma[1+(\mathrm{i}/kD)]}\frac{\sinh(\pi/kD)}{\cosh(\pi/kD)} \tag{4.8}$$

where we have replaced λ by $(\mathrm{i}kD)^{-1}$ and used hyperbolic instead of trigonometric functions.

In turn from (4.5*a*) we have that

$$T = \exp(-\pi/kD)/\cosh(\pi/kD). \tag{4.9}$$

We note though that before interpreting $|R|^2, |T|^2$ as the probabilities of reflection and transmission for the one-dimensional Coulomb potential, we must take into account the fact that the absolute values of the functions $W_{\pm\lambda}(\pm z)$ when $|z| \to \infty$ are not unity. In fact from (2.5*b*), and writing $z = 2\mathrm{i}k|x|$, we see that when $|z| = 2k|x| \to \infty$ we obtain [9]

$$W_\lambda(z) \to \exp\{-\mathrm{i}[k|x| + (kD)^{-1}\ln(2k|x|)\}\exp(\pi/2kD) \tag{4.10a}$$

$$W_{-\lambda}(-z) \to \exp\{\mathrm{i}[k|x| + (kD)^{-1}\ln(2k|x|)\}\exp(\pi/2kD) \tag{4.10b}$$

$$W_\lambda(-z) \to \exp\{\mathrm{i}[k|x| - (kD)^{-1}\ln(2k|x|)\}\exp(3\pi/2kD) \tag{4.10c}$$

and thus we have for $|z| \to \infty$ the following ratios for the absolute values

$$|W_{-\lambda}(-z)/W_\lambda(z)| = 1 \qquad |W_\lambda(-z)/W_\lambda(z)| = \exp(\pi/kD). \tag{4.11a, b}$$

If we normalize the incoming wave $W_\lambda(z)$ in (2.6*a*) to unit intensity, then from (4.11*a*) the reflected intensity is given by $|R|^2$, while from (4.11*b*) the transmitted one is

$$\exp(2\pi/kD)|T|^2 = [\cosh(\pi/kD)]^{-2}. \tag{4.12}$$

From (4.8) and (4.9) we then immediately check that

$$|R|^2 + \exp(2\pi/kD)|T|^2 = 1 \tag{4.13}$$

as required by the conservation of probability.

Note that when $k \to \infty$, $(\pi/kD) \to 0$ and so the probability (4.12) for transmission becomes almost 1. On the other hand if $k \to 0$, this probability tends to $4\exp(-2\pi/kD)$ and thus diminishes exponentially and, in fact, vanishes at $k = 0$.

The problem also has bound states characterized by the fact that there is no reflection i.e. for ks for which

$$R[(\mathrm{i}kD)^{-1}] = 0. \tag{4.14}$$

From (4.8) it is clear that this implies

$$(kD)^{-1} = in \qquad n = 0, 1, 2, \ldots \tag{4.15}$$

so the corresponding energies are real and negative, i.e.

$$E_n = -\frac{me^4Z^2}{2\hbar^2}\frac{1}{n^2}. \tag{4.16}$$

Note that the bound states, besides the standard ones of the hydrogen atom for $n = 1, 2, 3, \ldots$, include $E_0 = -\infty$, which is mentioned as a possibility for the one-dimensional Coulomb problem by several of the authors of [7].

Finally our main conclusion is that the one-dimensional Coulomb potential x^{-1} is penetrable, despite its divergence at $x = 0$.

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